CAPACITANCE السعـــــــــة

A capacitor is a device that stores energy; energy thus stored can either be associated with accumulated charge or it can be related to the stored electric field.

6.1 Capacitance Defined

Consider ***two conductors embedded in a homogeneous dielectric***. Conductor M_2 carries a total positive charge Q, and M_1 carries an equal negative charge. There are no other charges present, and the *total* charge of the system is zero.

Let us designate the potential difference between M_2 and M_1 as V_0. We may now define the *capacitance* of this two-conductor system as the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors

$$C = \frac{Q}{V_o}$$

In general terms, we determine Q by a surface integral over the positive conductors, and we find V_0 by carrying a unit positive charge from the negative to the positive surface,

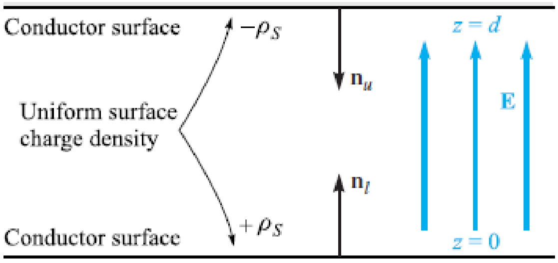
$$C = \frac{\oint \varepsilon E \, . \, dS}{-\int_{-}^{+} E \, . \, dL}$$

The capacitance is independent of the potential and total charge, for their ratio is constant. If the charge density is increased by a factor of N, Gauss's law indicates that the electric flux density or electric field intensity also increases by N, as does the potential difference. The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the homogeneous dielectric

Capacitance is measured in ***farads*** (F), where a farad is defined as one coulomb per volt

6.2 Parallel-Plate Capacitor

We can apply the definition of capacitance to a simple two-conductor system in which the conductors are identical, infinite parallel planes with separation d (Figure below). Choosing the lower conducting plane at z = 0 and the upper one at z = d, a uniform sheet of surface charge ρ_s on each conductor leads to the uniform field



$$\boldsymbol{E} = \frac{\rho_s}{\varepsilon}\mathbf{a}_z$$

where the permittivity of the dielectric is ε ,

$$\boldsymbol{D} = \rho_s\mathbf{a}_z$$

The potential difference between lower and upper planes is

$$V_o = -\int_{upper}^{lower} \boldsymbol{E}\,.\,dL = -\int_{d}^{0} \frac{\rho_s}{\varepsilon}dz = \frac{\rho_s}{\varepsilon}d$$

Since the total charge on either plane is infinite, the capacitance is infinite. A more practical answer is obtained by considering planes, each of area S, whose linear dimensions are much greater than their separation d.

$$Q = \rho_s\, S$$

$$C = \frac{Q}{V_o} = \frac{\rho_s\, S}{\frac{\rho_s}{\varepsilon}d}$$

$$C = \frac{\varepsilon\, S}{d}$$

Example: Calculate the capacitance of a parallel-plate capacitor having a mica dielectric, ε_r= 6, a plate area of 10 cm^2, and a separation of 0.01 cm.?

Solution:

$$C = \frac{\varepsilon\, S}{d} = \frac{\varepsilon_0\varepsilon_r\, S}{d} = \frac{8.85 \times 10^{-12} \times 6 \times 10}{0.01} = 53.1\, nF$$

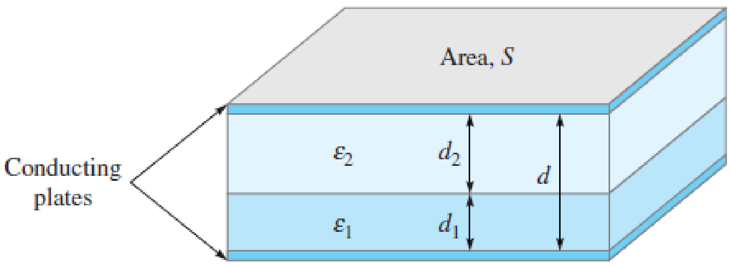
6.3 Multiple-Dielectric Capacitors

When two dielectrics are present in a capacitor with the interface ***normal*** to E and D, as shown in Figure below, the equivalent capacitance can be obtained by treating the arrangement as two capacitors in series

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 * C_2}{C_1 + C_2}$$

$$C_1 = \frac{\varepsilon_1 S}{d_1} ,$$

$$C_2 = \frac{\varepsilon_2 S}{d_2}$$

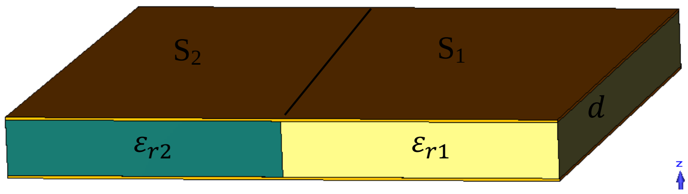


When two dielectrics are present in a capacitor with the interface ***parallel*** to E and D, as shown in Figure below, the equivalent capacitance can be obtained by treating the arrangement as two capacitors in parallel

$$C = C_1 + C_2$$

$$C_1 = \frac{\varepsilon_1 S_1}{d}$$

$$C_2 = \frac{\varepsilon_2 S_2}{d}$$



Example: A parallel-plate capacitor with area 0.30 m^2 and separation 5.5 mm contains three dielectrics with interfaces normal to E and D, as follows: $\varepsilon_{r1} = 3$, d_l = 1mm; $\varepsilon_{r2} = 4$, d_2 = 2mm; $\varepsilon_{r3} = 6$, d_3 = 2.5 mm. Find the capacitance?

Solution:

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_1 = \frac{\varepsilon_1\ \mathrm{S}}{d_1} = \frac{\varepsilon_{r1}\varepsilon_0 S}{d_1}$$

$$C_1 = \frac{3 \times 8.85 \times 10^{-12} \times 0.3}{10^{-3}} = 7.96\ nF$$

$$C_2 = \frac{\varepsilon_2\ \mathrm{S}}{d_2} = \frac{\varepsilon_{r2}\varepsilon_0 S}{d_2}$$

$$C_2 = \frac{4 \times 8.85 \times 10^{-12} \times 0.3}{2 \times 10^{-3}} = 5.31\ nF$$

$$C_3 = \frac{\varepsilon_3\ \mathrm{S}}{d_3} = \frac{\varepsilon_{r3}\varepsilon_0 S}{d_3}$$

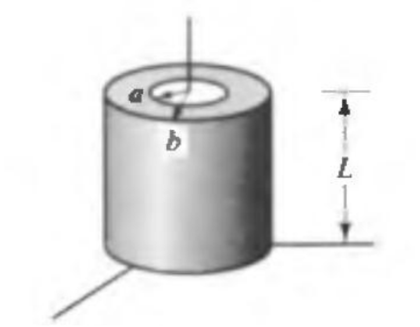
$$C_3 = \frac{6 \times 8.85 \times 10^{-12} \times 0.3}{10^{-3}} = 6.37\ nF$$

$$\frac{1}{C_T} = \frac{1}{7.96 \times 10^{-9}} + \frac{1}{5.31 \times 10^{-9}} + \frac{1}{6.37 \times 10^{-9}}$$

$$C_T = 2.12\ nF$$

6.4 Capacitance of coaxial cable or coaxial capacitor

We choose a coaxial cable or coaxial capacitor of inner radius a, outer radius b, and length L



$$D = \frac{\rho_s a}{r}\ \mathrm{a}_r$$

$$E = \frac{\rho_s a}{\varepsilon\, r}\ \mathrm{a}_r$$

$$V_{ab} = -\int_b^a \frac{\rho_s a}{\varepsilon\, r}\mathrm{a}_r . dr\mathrm{a}_r$$

and the voltage difference between the conductors is

$$V_{ab} = \frac{\rho_s a}{\varepsilon}\ln\frac{b}{a}$$

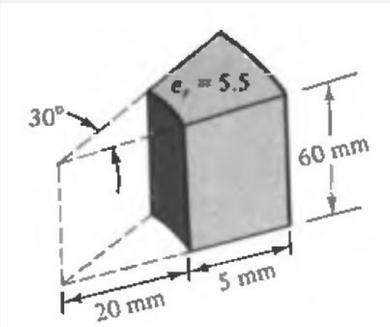
The total charge on the inner conductor is

$$Q = \rho_s(2\pi a L)$$

$$C = \frac{Q}{V}$$

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

Example: Find the capacitance between the inner and outer curved conductor surfaces shown in Figure below?



Solution:

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)} \quad for\ coaxil\ capacitor$$

$$\therefore C = \frac{\alpha}{2\pi}\ \frac{2\pi\varepsilon L}{\ln(b/a)} \quad for\ a\ part\ of coaxil\ capacitor$$

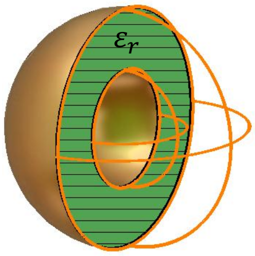
$$C = \frac{\alpha}{1}\ \frac{\varepsilon L}{\ln(b/a)} = \frac{30^0 \times \frac{\pi}{180}}{1}\ \frac{5.5 \times 8.85 \times 10^{-12} \times 60 \times 10^{-3}}{\ln(25/20)} = 6.86\ PF$$

6.5 Spherical Capacitor

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b, $b > a$. The expression for the electric field was obtained previously by Gauss's law

$$E = \frac{Q}{4\pi\varepsilon r^2}\mathrm{a}_r$$

where the region between the spheres is a dielectric with permittivity ε . The expression for potential difference was found from this by the line integral.



Thus,

$$V_{ab} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Here Q represents the total charge on the inner sphere, and the capacitance becomes

$$C = \frac{Q}{V}$$

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

If we allow the outer sphere to become infinitely large, we obtain the capacitance of an isolated spherical conductor

$$C = 4\pi\varepsilon a$$

Example: Find the capacitance between the inner (r=2) and outer (r=3) sphere conductor surfaces if $\varepsilon_{\mathrm{r}} = 2.5$?

Solution:

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times 8.85 \times 10^{-12} \times 2.5}{\frac{1}{2} - \frac{1}{3}} = 1.66\, nF$$

6.6 Energy Stored in a Capacitor

The total energy stored in the capacitor is

$$W_E = \frac{1}{2}\int_{vol} \varepsilon\, E^2 dv = \frac{1}{2}\int_0^S \int_d^0 \frac{\varepsilon \rho_s^2}{\varepsilon^2} dz dS = \frac{1}{2}\frac{\rho_s^2}{\varepsilon} Sd = \frac{1}{2}\frac{\varepsilon S}{d}\frac{\rho_s^2 d^2}{\varepsilon^2}$$

$$W_E = \frac{1}{2} C V_o^2 = \frac{1}{2} Q V_o = \frac{1}{2}\frac{Q^2}{C}$$

Example: Find the relative permittivity of the dielectric material present in a parallel-plate capacitor if: (*a*) $S = 0.12\text{m}^2$, $d = 80\ \mu\text{m}$, $V_0 = 12$ V, and the capacitor contains 1μJ of energy; (*b*) the stored energy density is 100 J/m^3, $V_0 = 200$ V, and $d = 45\mu$m; (*c*) $E = 200$ kV/m and $\rho_s = 20\mu\text{C/m}^2$?

Solution:

(***a***) $W_E = \frac{1}{2} C V_o^2$

$$1 \times 10^{-6} = \frac{1}{2} C (12)^2 \qquad \Rightarrow\ C = 13.88\ nF$$

$$C = \frac{\varepsilon\, S}{d} = \frac{\varepsilon_r \varepsilon_0\, S}{d}$$

$$\varepsilon_r = \frac{C\ d}{\varepsilon_0\ S} = \frac{13.88 \times 10^{-9} \times 80 \times 10^{-6}}{8.85 \times 10^{-12} \times 0.12} = 1.05$$

(***b***) $W = 100 \times 0.12 \times 45 \times 10^{-6} = 540\ \mu\text{J}$

$$C = \frac{2W}{{V_0}^2} = \frac{2 \times 540 \times 10^{-6}}{(200)^2} = 27\ nF$$

$$\varepsilon_r = \frac{C\ d}{\varepsilon_0\ S} = \frac{27 \times 10^{-9} \times 80 \times 10^{-6}}{8.85 \times 10^{-12} \times 0.12} = 1.14$$

(***c***) $E = \frac{\rho_s}{\varepsilon_r \varepsilon_0} \qquad \Longrightarrow \qquad \varepsilon_r = \frac{\rho_s}{\varepsilon_0\ E} = \frac{20 \times 10^{-6}}{8.85 \times 10^{-12} \times 200 \times 10^3}$

6.7 Poisson's and Laplace's Equations

An alternate approach would be to start with known potentials on each conductor, and then work backward to find the charge in terms of the known potential difference. The capacitance in either case is found by the ratio *Q/V*.

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law

$$\nabla . D = \rho_v$$

$$E = -\nabla V$$

by substitution we have

$$\nabla . D = \nabla .(\varepsilon E) = -\nabla .(\varepsilon \nabla V) = \rho_v$$

$$\nabla . \nabla V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

Equation above is ***Poisson's equation***,

If ρ_v = 0, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0$$

which is ***Laplace's equation***. The ∇^2 operation is called the *Laplacian of V*.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad \text{(Cartesian)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \emptyset^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \emptyset^2} \qquad \text{(spherical)}$$

6.8 Examples of The Solution of Laplace's Equation

Several methods have been developed for solving Laplace's equation. The simplest method is that of direct integration. We will use this technique to work several examples involving ***one-dimensional*** potential variation in various coordinate systems in this section.

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given V, use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$.

2. Use $\mathbf{D} = \varepsilon\mathbf{E}$ to find $\mathbf{D}$.

3. Evaluate $\mathbf{D}$ at either capacitor plate, $\mathbf{D} = \mathbf{D}S = D_N\,\mathbf{a}_N$.

4. Recognize that $\rho_S = D_N$.

5. Find Q by a surface integration over the capacitor plate, $Q = \int_S \rho_s \, dS$

- ***V is a function only of x:***

Laplace's equation reduces to

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} = 0$$

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

$$\frac{d^2 V}{dx^2} = 0$$

We integrate twice, obtaining

$$\frac{dV}{dx} = A$$

$$V = Ax + B$$

where A and B are constants of integration

Example: Find the capacitance of a parallel-plate capacitor of plate area S, plate separation d, and potential difference V_0 between plates?

Solution:

$$\frac{d^2 V}{dz^2} = 0$$

$$V = Az + B$$

$V = 0 \; at \; Z = 0$

$0 = A(0) + B \qquad , \quad \therefore \; B = 0$

$V = Az$

$V = V_0 \;\; at \; Z = d$

$V_0 = Ad \qquad ,\therefore \; A = \frac{V_0}{d}$

$V = \frac{V_0}{d} z$

$E = -\nabla V = -\frac{\partial V}{\partial z} \mathrm{a}_z = -\frac{V_0}{d} \mathrm{a}_z$

$D = \varepsilon E = -\frac{\varepsilon V_0}{d} \mathrm{a}_z$

$\rho_s = D_N = -\frac{\varepsilon V_0}{d}$

$Q = \int \rho_s \, ds = \int -\frac{\varepsilon V_0}{d} \; ds = -\frac{\varepsilon V_0 S}{d}$

$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$

- ***V as a function of ρ***

Laplace’s equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Noting the ρ in the denominator, we exclude $\rho = 0$ from our solution and then multiply by ρ and integrate

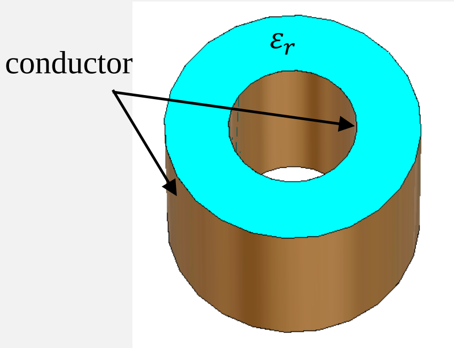
$$\rho \frac{dV}{d\rho} = A$$

where a total derivative replaces the partial derivative because V varies only with ρ. Next, rearrange, and integrate again,

$$V = A \ln \rho \; + B$$

Example: Find the capacitance of the coaxial capacitor or coaxial transmission line?

Solution:



$$V = A\ln\rho + B$$

$$V = 0 \quad at\ \rho = b$$

$$0 = A\ln b + B$$

$$\therefore\ B = -A\ln b$$

$$V = A\ln\rho - A\ln b = A\ln\frac{\rho}{b}$$

$$V = V_0 \quad at\ \rho = a$$

$$V_0 = A\ln\frac{a}{b}$$

$$\therefore\ A = \frac{V_0}{\ln\frac{a}{b}}$$

$$V = \frac{V_0\ln\frac{\rho}{b}}{\ln\frac{a}{b}}$$

$$E = -\nabla V = \frac{V_0}{\rho}\frac{1}{\ln\frac{b}{a}}\,\mathrm{a}_\rho$$

$$D = \varepsilon E = \frac{\varepsilon V_0}{\rho}\frac{1}{\ln\frac{b}{a}}\,\mathrm{a}_\rho$$

$$\rho_s = D_N = \frac{\varepsilon V_0}{\rho}\frac{1}{\ln\frac{b}{a}}\,\mathrm{a}_\rho$$

$$Q = \int \rho_s\, ds$$

$$Q = \int_0^L\int_0^{2\pi}\frac{\varepsilon V_0}{\rho}\frac{1}{\ln\frac{b}{a}}\ \rho\, d\emptyset\, dz$$

$$Q = \frac{2\pi L\varepsilon V_0}{\ln\frac{b}{a}}$$

$$C = \frac{Q}{V_0} = \frac{2\pi L\varepsilon}{\ln\frac{b}{a}}$$

- ***V as a function of ∅ in cylindrical coordinates***

Laplace's equation is now

$$\frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \emptyset^2}\right) = 0$$

We exclude ρ = 0 and have

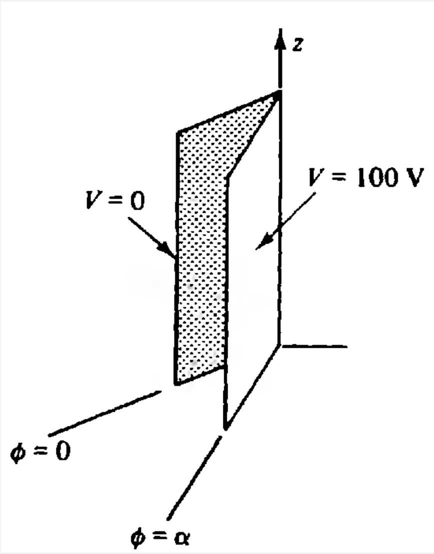
$$\frac{d^2 V}{d\emptyset^2} = 0$$

The solution is

$$V = A\emptyset + B$$

Example: In cylindrical coordinates two ∅ = const, planes are insulated along the z axis, as shown in Fig. 8-9. Neglect fringing and find the expression for E between the planes, assuming a potential of 100 V for ∅ = a, and a zero reference at ∅ = 0?

Solution:



$V = A\emptyset + B$

$V = 0 \quad at\ \emptyset = 0$

$0 = A(0) + B$

$\therefore\ B = 0$

$V = A\emptyset$

$V = 100 \quad at \quad \emptyset = \alpha$

$100 = A\alpha$

$$A = \frac{100}{\alpha}$$

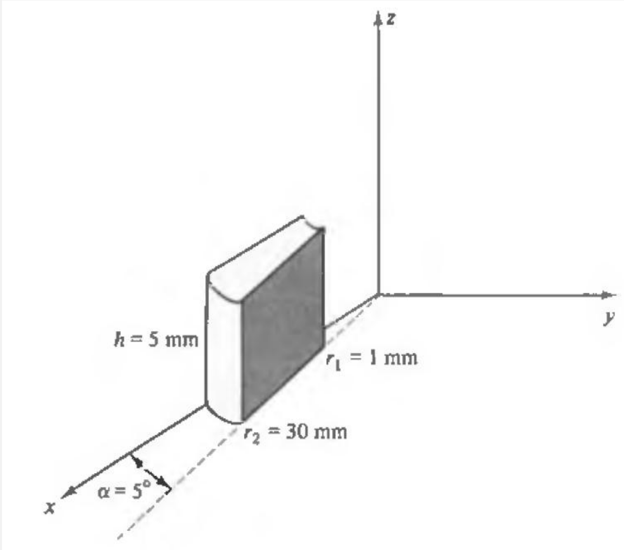
$$V = \frac{100}{\alpha}\emptyset$$

$$E = -\nabla V = \frac{-1}{\rho}\frac{\partial v}{\partial \emptyset}\ \mathrm{a}_{\emptyset}$$

$$E = \frac{-1}{\rho}\frac{\partial}{\partial \emptyset}\left(\frac{100}{\alpha}\emptyset\right)\mathrm{a}_{\emptyset} = \frac{-100}{\rho\alpha}\mathrm{a}_{\emptyset}$$

Example: In the capacitor shown in Figure below, the region between the plates is filled with a dielectric having ε_r= 4.5. Find the capacitance?

Solution:



From example above

$$E = \frac{-100}{\rho\alpha} a_\phi$$

$$D = \varepsilon E = \frac{-100\varepsilon}{\rho\alpha} a_\phi$$

$$\rho_s = D = \frac{-100\varepsilon}{\rho\alpha} a_\phi$$

$$Q = \int \rho_s ds = \int_0^{0.005} \int_{0.001}^{0.03} \frac{-100\varepsilon}{\rho\alpha}\, d\rho\, dz$$

$$Q = \frac{100\varepsilon}{\alpha} \times 0.005 \ln\frac{30}{1}$$

$$C = \frac{Q}{V} = \frac{Q}{100} = \frac{0.017\varepsilon}{\alpha} = \frac{0.017 \times 4.5 \times 8.85 \times 10^{-12}}{5 \times \pi/180} = 7.76\ PF$$

- ***V as a function of θin Spherical coordinates***

Laplace's equation is now

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

We exclude r = 0 and θ = 0 or π and have

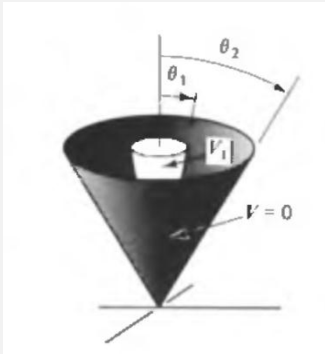
$$\sin\theta \frac{dV}{d\theta} = A$$

The second integral is then

$$V = \int \frac{A}{\sin\theta} d\theta + B$$

$$V = A \ln\left(\tan\frac{\theta}{2}\right) + B$$

Example: Solve Laplace's equation for the region between coaxial cones, as shown in Figure below. A potential V, is assumed at θ_1, and V = 0 at θ_2. The cone vertices are insulated at r = 0? (b) let θ_1 = 10°, θ_2=30°, and V_1 = 100V. Find the voltage at θ = 20°, At what angle θ is the voltage 50 V?



Solution:

$$V = A\ln\left(\tan\frac{\theta}{2}\right) + B$$

$$V = 0 \quad at \quad \theta = \theta_2$$

$$0 = A\ln\left(\tan\frac{\theta_2}{2}\right) + B \qquad , \quad \therefore\ B = -A\ln\left(\tan\frac{\theta_2}{2}\right)$$

$$V = A\ln\left(\tan\frac{\theta}{2}\right) - A\ln\left(\tan\frac{\theta_2}{2}\right)$$

$$V = V_1 \quad at \quad \theta = \theta_1$$

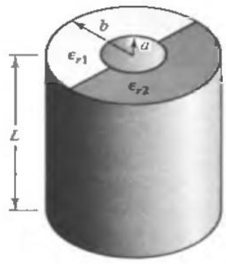
$$V_1 = A\ln\left(\tan\frac{\theta_1}{2}\right) - A\ln\left(\tan\frac{\theta_2}{2}\right) = A\left[\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)\right]$$

$$\therefore A = \frac{V_1}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}$$

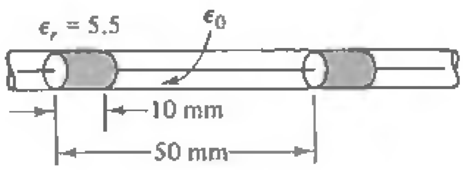
$$V = V_1\ \frac{\ln\left(\tan\frac{\theta}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}$$

Home work

Q_1: In the cylindrical capacitor shown in Fig. below each dielectric occupies one-half the volume. Find the capacitance?



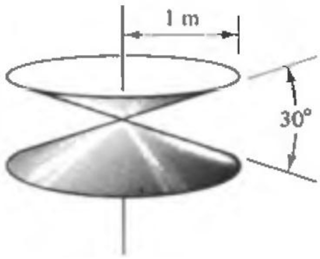
Q_2: The coaxial cable in Fig. below has an inner conductor radius of 0.5 mm and an outer conductor radius of 5 mm. find the capacitance per unit length with spacers as shown.



Ans: 45.9 PF/m

Q_3: A parallel-plate capacitor has its dielectric changed from ε_{r1} = 2.0 to ε_{r2} = 6.0. It is noted that the stored energy remain fixed: $W_2 = W_1$. Examine the changes, if any, in V, C, D, E, Q, and ρ_s?

Q_4: Find the capacitance between the two cones of Fig. below. Assume free space?



Ans: $12.28\varepsilon_0$

Q_5: A parallel-plate capacitor of 8nF has an area 1.51 m^2 and separation 10 mm. What separation would be required to obtain the same capacitance with free space between the plates?

Ans: 1.67 mm